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On the maximum drag reduction due to added polymers in Poiseuille flow

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The addition of elastic polymers to turbulent liquids is known to produce significant drag reduction. In this study, we prove that the drag in pipe and channel flows of an unforced laminar fluid constitutes a lower bound for the drag of a fluid containing dilute elastic polymers. Further, the addition of elastic polymers to laminar fluids invariably increases drag. This proof does not rely on the adoption of a particular constitutive equation for the polymer force, and would also be applicable to other similar methods of drag reduction, which are also achieved by the addition of certain particles to a flow. Examples of such methods include the addition of surfactants to a flowing liquid and the presence of sand particles in sandstorms and water droplets in cyclones.

Key words: drag reduction

1. Introduction

The volume flux of a laminar Poiseuille flow, in a channel or pipe, will always be greater than the volume flux of the equivalent turbulent flow (Thomas 1942). (In this paper, two flows are considered 'equivalent' if they are driven by the same average pressure gradient.) However, various methods have been developed by which the volume flux of a turbulent flow can be increased. These methods range from adding riblets to the wall surface (Karniadakis & Choi 2003), to targeted blowing and suction at the flow boundary (Collis *et al.* 2004) (a form of flow-control known as transpiration). In this study, we consider drag reduction due to the presence of elastic polymers in a turbulently flowing liquid.

It has been well demonstrated that adding elastic polymers to a turbulent liquid will dramatically reduce the drag on the fluid, and significantly increase the flow rate (Toms 1948) (see White & Mungal 2008 for a recent review). It has been argued that the polymers reduce drag by transporting momentum within the fluid (Min *et al.* 2003), as well as by countering vorticity and eddying motions, and thereby reducing fluctuations (Luchik & Tiederman 1988; Kim *et al.* 2007).

Experimental results show that at critical values of polymer-concentration and polymer-relaxation time, the drag experienced by the fluid will be reduced, and that

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this effect can require only very low concentrations of the polymer (Virk *et al.* 1967; Sreenivasan & White 2000). In one particular example, a concentration in the order of 10 p.p.m. of a certain long-chain polymer has been found to be capable of reducing drag by up to 80 % (Virk 1975). However, a limit appears to exist to the drag reduction that an added polymer may produce. This limit is known as Virk's asymptote (Virk 1971). It, however, remains a purely empirical result, and there is no theoretical proof that it constitutes the maximum allowable drag reduction.

In this paper, we prove that the force caused by the presence of the polymer will be incapable of raising the bulk flow rate of a turbulent Poiseuille flow to beyond that of the equivalent unforced laminar flow. (Throughout this paper, the term 'unforced' is used to describe a flow which is not subjected to any body forces.) We prove this without reference to any constitutive equation for the polymer force. We also prove that the polymer force, when acting upon a laminar Poiseuille flow, will decrease the fluid's bulk flow rate. This is not be true for all methods of drag reduction. In the case of transpiration flow-control, it has been shown using direct numerical simulations of low-Reynolds-number turbulent flows that it is possible to increase the volume flux of a turbulent fluid to beyond that of the equivalent laminar flow (Min *et al.* 2006). Moreover, the conditions required to produce sublaminar drag in a turbulent Poiseuille flow through a channel have been derived, for arbitrary Reynolds number, by Marusic, Joseph & Mahesh (2007). Subsequently, Bewley (2009) showed that the power saved by reducing drag to sublaminar levels via blowing and suction-flow control must be less than the power transferred to the fluid by that flow control.

A caveat must be made about the proof presented in this paper. The presence of elastic polymers in a Newtonian liquid generally causes the solution to become significantly shear-thinning. However, if the polymer is sufficiently dilute, then the fluid's viscosity will be only negligibly affected by the polymer's presence, and the fluid will remain effectively Newtonian. Such fluids are also known as Boger fluids, and their advantage over other polymer-containing fluids is that they allow the effects of the polymer's elasticity to be considered separately from their shear-thinning effects (James 2009).

In the subsequent proof, it has been assumed that the fluid's density and viscosity are not significantly affected by the presence of the polymer. However, it must be noted that the presence of the polymer increases the viscosity of the solution beyond that of the pure solvent, and that a stress applied to the solution will merely act to reduce this increase. Thus, regardless of the applied stress, the viscosity of the solution should remain greater than that of the pure Newtonian solvent.

It should, therefore, be reasonable to assume that if the addition of the polymer cannot produce sublaminar drag when the increase in viscosity is neglected, then neither can it when the increase in viscosity is not neglected. This follows from the fact that a higher viscosity will impede flow.

The polymers may also affect the fluid in a more fundamental way, by either delaying or facilitating the onset of turbulence. When undergoing rotational flow, such as Taylor–Couette flow, fluids containing polymers have been found to transition to turbulence at lower Reynolds numbers than their pure solvents (Shaqfeh 1996; Groisman & Steinberg 2000). Conversely, the presence of polymers in pipe flows has been found to actually delay the onset of turbulence in a small minority of cases (Virk *et al.* 1967).

Other forms of passive drag reduction also exist, to which the proof presented herein (and its caveat) will apply. The presence of aggregates of surfactant molecules, known as micelles, in a turbulently flowing liquid has been known to be capable of producing a slightly greater degree of drag reduction than the presence of elastic polymers (Warholic, Schmidt & Hanratty 1999). A similar phenomenon is also believed to occur in sandstorms (Gore & Crowe 1989) and cyclones (Barenblatt, Chorin & Prostokishin 2005). Under certain circumstances, strong turbulent winds suspending sand particles or water droplets can encounter significantly less drag than the equivalent flows of pure turbulent air.

What each of these methods of drag reduction has in common with polymer drag reduction is that they result from the interaction of solutes with the flow, and involve no external source of energy or momentum.

2. Equations of channel flow

The proof is presented in § 4 of this paper. It is an extension of the work by Busse (1970) and Howard (1972), who considered the minimum drag for the flow of an unforced fluid in a channel. The derivation follows a similar path to that presented by Marusic *et al.* (2007) for a Poiseuille flow subjected to blowing and suction-flow control. This section contains the mathematical preliminaries, which we make use of in § 4 and in an alternative proof presented in § 5.

We consider an incompressible fluid driven by a constant pressure gradient and subjected to a force per unit volume $\boldsymbol{f}$ caused by the presence of the polymer. In this study, time and velocity have been normalised using ν, the kinematic viscosity of the fluid, and d, the height of the channel. The pressure and polymer force have also been normalised by the fluid's density ρ. The resulting Navier–Stokes and continuity equations are written as follows:

$$\frac{\partial \boldsymbol{V}}{\partial t} + \boldsymbol{V}\cdot\nabla\boldsymbol{V} = -\nabla p + \boldsymbol{e}_x P + \nabla^2\boldsymbol{V} + \boldsymbol{f}, \tag{2.1}$$

$$\nabla\cdot\boldsymbol{V} = 0, \tag{2.2}$$

where $\boldsymbol{V}$ denotes the normalised velocity. The pressure has been split between a constant pressure gradient P, caused by an imposed pressure gradient, and a variable pressure function p, such that the total pressure at any point is given by

$$p_{total}(x, y, z, t) = p(x, y, z, t) - Px. \tag{2.3}$$

Hence, the x direction is the direction of the imposed pressure gradient, and is, therefore, also the streamwise direction. The y direction is defined as the spanwise direction, and the z direction is defined as the wall-normal direction.

The system has been defined as a channel of infinite length and width, and unit height. A diagram of the channel is shown in figure 1. For the purposes of the subsequent derivations, we define the channel as having length and width L, and we consider the limiting case as $L \to \infty$. The Cartesian coordinate system has been used, and the domain is, therefore,

$$-\infty < x, y < \infty, \quad -\tfrac{1}{2} \leqslant z \leqslant \tfrac{1}{2}. \tag{2.4}$$

The subsequent derivations were also performed for a pipe flow, the results of which can be found in the Appendix. For all quantities, wall-parallel averages are denoted by an overbar and are defined via

$$\bar{F}(z, t) \stackrel{\text{def}}{=} \lim_{L\to\infty} \frac{1}{L^2} \int_{-L/2}^{L/2} \int_{-L/2}^{L/2} F(x, y, z, t)\,\mathrm{d}x\,\mathrm{d}y. \tag{2.5}$$

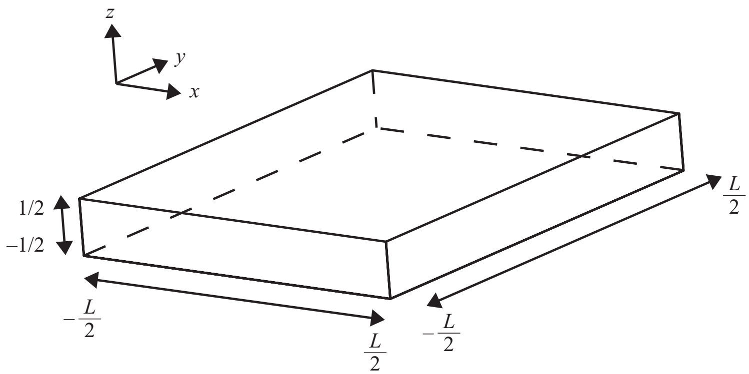

FIGURE 1. Diagram of the channel domain. We consider an infinite channel in which $L \to \infty$.

It should be noted that for a statistically steady-state flow, a wall-parallel average will be equivalent to a temporal average. This fact will be used subsequently. In this paper, a statistically steady-state flow refers to a fully developed flow, whose bulk (spatially averaged) characteristics are independent of time. This will not necessarily be a flow in which the local velocity at any point is independent of time.

An average over the entire channel is denoted by angled brackets and is defined via

$$\langle F(t) \rangle \stackrel{\text{def}}{=} \int_{-1/2}^{1/2} \bar{F}(z,t)\,\mathrm{d}z. \tag{2.6}$$

The Reynolds number, based on the channel height and the bulk velocity, is $Re_B = \langle \bar{V}_x \rangle$. We also decompose the velocity, pressure and polymer force into wall-parallel averaged components (referred to from this point on as 'mean' components) and fluctuating components,

$$\boldsymbol{V} = \bar{\boldsymbol{V}} + \boldsymbol{u}, \quad p = \bar{p} + p', \quad \boldsymbol{f} = \bar{\boldsymbol{f}} + \boldsymbol{f}', \tag{2.7}$$

where the mean of each fluctuation is zero: $\bar{\boldsymbol{u}} = \overline{\boldsymbol{f}'} = \boldsymbol{0}$, $\overline{p'} = 0$. We assume the flow is subjected to a no-slip boundary condition, and so we have,

$$\bar{\boldsymbol{V}} = \boldsymbol{u} = \boldsymbol{0}, \quad \text{at } z = \pm\tfrac{1}{2}. \tag{2.8}$$

Since the x direction is the direction of the constant pressure gradient P, and since there exists no driving force which should sustain a bulk flow in the y or z directions, the velocity functions may be written as

$$\bar{\boldsymbol{V}}(z,t) = (\bar{V}_x, 0, 0), \quad \boldsymbol{u}(x,y,z,t) = (u, v, w). \tag{2.9}$$

2.1. *Energy equations*

We derive energy equations relating to the mean and fluctuating components of a statistically steady-state flow. These will be used in subsequent equations.

To do this, we first decompose all terms in the Navier–Stokes equation into their mean and fluctuating components, and then making use of (2.2), we have

$$\frac{\partial \bar{\boldsymbol{V}}}{\partial t} + \frac{\partial \boldsymbol{u}}{\partial t} + \nabla\boldsymbol{\cdot}(\bar{\boldsymbol{V}}\boldsymbol{u} + \boldsymbol{u}\bar{\boldsymbol{V}} + \boldsymbol{u}\boldsymbol{u}) = -\boldsymbol{e}_z\frac{\partial \bar{p}}{\partial z} - \nabla p' + e_x P + \frac{\partial^2 \bar{\boldsymbol{V}}}{\partial z^2} + \nabla^2\boldsymbol{u} + \bar{\boldsymbol{f}} + \boldsymbol{f}'. \tag{2.10}$$

The wall-parallel average of (2.10) is

$$\frac{\partial \bar{\boldsymbol{V}}}{\partial t} + \frac{\partial}{\partial z}(\overline{\boldsymbol{u}\boldsymbol{u}}) = -e_z \frac{\partial \bar{p}}{\partial z} + e_x P + \frac{\partial^2 \bar{\boldsymbol{V}}}{\partial z^2} + \bar{\boldsymbol{f}}. \tag{2.11}$$

The evolution equation for the fluctuations in velocity is, therefore, the difference (2.10)–(2.11), which is

$$\frac{\partial \boldsymbol{u}}{\partial t} + \nabla \cdot (\bar{\boldsymbol{V}}\boldsymbol{u} + \boldsymbol{u}\bar{\boldsymbol{V}} + \boldsymbol{u}\boldsymbol{u} - \overline{\boldsymbol{u}\boldsymbol{u}}) = -\nabla p' + \nabla^2 \boldsymbol{u} + \boldsymbol{f}'. \tag{2.12}$$

The energy equation for the mean flow can be found by taking $\langle \bar{V} \cdot (2.11) \rangle$. We may use the fact that for a statistically steady-state flow, a wall-parallel average is equivalent to a temporal average to remove the time-derivative term form this energy equation. We thus obtain,

$$-\left\langle \overline{uw} \frac{\mathrm{d}\bar{V}_x}{\mathrm{d}z} \right\rangle = P \langle \bar{V}_x \rangle - \left\langle \left| \frac{\mathrm{d}\bar{V}_x}{\mathrm{d}z} \right|^2 \right\rangle + \langle \bar{\boldsymbol{V}} \cdot \bar{\boldsymbol{f}} \rangle. \tag{2.13}$$

Similarly, the energy equation for the fluctuations in the flow can be found by taking $\langle u \cdot (2.12) \rangle$. Again, removing the time-derivative term form the resulting equation gives,

$$\left\langle \overline{uw} \frac{\mathrm{d}\bar{V}_x}{\mathrm{d}z} \right\rangle = -\langle |\nabla \boldsymbol{u}|^2 \rangle + \langle \boldsymbol{u} \cdot \boldsymbol{f}' \rangle. \tag{2.14}$$

The overall energy equation is then simply the sum of (2.14) and (2.15), which is

$$0 = P \langle \bar{V}_x \rangle - \left\langle \left| \frac{\mathrm{d}\bar{V}_x}{\mathrm{d}z} \right|^2 \right\rangle - \langle |\nabla \boldsymbol{u}|^2 \rangle + \langle \boldsymbol{V} \cdot \boldsymbol{f} \rangle. \tag{2.15}$$

The term $\langle \boldsymbol{V} \cdot \boldsymbol{f} \rangle$ in (2.15) represents the rate at which the polymer does work upon the fluid per unit of volume, throughout the channel. If negative, $\langle \boldsymbol{V} \cdot \boldsymbol{f} \rangle$ represents the rate at which the polymer extracts energy from the flow.

3. Work done by the polymer upon the flow

In the previous section, we showed that $\langle \boldsymbol{V} \cdot \boldsymbol{f} \rangle$, which represents the overall rate at which the polymer does work upon the flow per unit of volume, is crucial to determining the bulk flow rate of the fluid $\langle \bar{V}_x \rangle$. In this section, we show that $\langle \boldsymbol{V} \cdot \boldsymbol{f} \rangle < 0$.

The exact physical causes of drag reduction due to elastic polymers are open to debate. As mentioned in § 1 of this paper, they are believed to involve fundamentally altering the turbulent fluid's flow profile by transporting momentum within the fluid, and by countering vorticity and eddying motions. However, none of these mechanisms involves a net transfer of energy from the polymer to the fluid.

The polymer draws energy from the flow, and may return energy to the flow. The energy is stored meanwhile as elastic energy within the polymer. Where the term $\boldsymbol{V} \cdot \boldsymbol{f}$ is positive, the polymer is imparting energy to the flow; and where it is negative, the polymer is drawing energy from the flow. The polymer has no source of energy apart from the flow, and therefore, the overall work done by the polymer upon the flow $\langle \boldsymbol{V} \cdot \boldsymbol{f} \rangle$, cannot be positive.

There are two paths by which the presence of the polymer may affect the total energy of the flow: the first is the transport of energy into and out of the channel as elastic energy due to the stretching of the polymer molecules, and the second is the dissipation of elastic energy from within the polymer molecules. However, in a statistically steady-state flow, the average elongation of a polymer molecule entering the channel will be equal to the average elongation exiting the channel. Hence, there will in fact be no net transport of elastic energy into, or out of, the channel.

Thus, since the overall work done by the polymer upon the flow must be equal to the rate of dissipation of elastic energy within the polymer molecules, we may say that,

$$\langle \boldsymbol{V} \cdot \boldsymbol{f} \rangle = -\langle \epsilon_p \rangle \leqslant 0, \tag{3.1}$$

where $\epsilon_p(x, y, z, t)$ denotes the rate at which elastic energy from within the polymer molecules is dissipating at a point in the flow. For a discussion of the magnitude of this dissipation rate, see Ptasinski *et al.* (2003).

4. Volume-flux comparison between laminar and turbulent flows

The flow profile for an unforced laminar fluid is given by

$$U_l = \frac{P}{2}\left(\frac{1}{4} - z^2\right), \tag{4.1}$$

and its bulk flow rate will be

$$\langle U_l \rangle = \frac{P}{12}. \tag{4.2}$$

We now consider a turbulent flow subjected to a polymer force. To this end, we redefine the polymer force in terms of a stress tensor

$$\nabla \cdot \boldsymbol{\tau} = \boldsymbol{f}. \tag{4.3}$$

This tensor refers to the stress experienced by the fluid due to the polymer force. For a statistically steady-state flow, the component of (2.11) in the x dimension may now be written as

$$\frac{\mathrm{d}}{\mathrm{d}z}\left[\overline{uw} - Pz - \frac{\mathrm{d}\bar{V}_x}{\mathrm{d}z} - \bar{\tau}_{xz}\right] = 0, \tag{4.4}$$

where $\bar{\tau}_{xz}$ is one component of the mean of the stress tensor, and is defined by

$$\frac{\mathrm{d}\bar{\tau}_{xz}}{\mathrm{d}z} = \bar{f}_x. \tag{4.5}$$

We can obtain an expression for P by integrating (4.4), which results in

$$Pz = \overline{uw} - \langle \overline{uw} \rangle - \bar{\tau}_{xz} + \langle \bar{\tau}_{xz} \rangle - \frac{\mathrm{d}\bar{V}_x}{\mathrm{d}z}. \tag{4.6}$$

The bulk flow rate can, therefore, be found by taking $\langle z \cdot (4.6) \rangle$ and rearranging. By doing so, we obtain

$$\langle \bar{V}_x \rangle = \frac{P}{12} - \langle z\overline{uw} \rangle + \langle z\bar{\tau}_{xz} \rangle. \tag{4.7}$$

The $\langle z\overline{uw} \rangle$ term in the above equation can be evaluated by taking $\langle \overline{uw} \cdot (4.6) \rangle$ and substituting the energy equation for the fluctuations (2.14). The $\langle z\bar{\tau}_{xz} \rangle$ term can be evaluated similarly by taking $\langle \bar{\tau}_{xz} \cdot (4.6) \rangle$, and integrating one of the resulting terms by parts, taking into account the no-slip boundary condition and (4.5). In doing so, we obtain a new equation for $\langle \bar{V}_x \rangle$ after substituting the results into (4.7). Then by comparing this result to (4.2), we obtain the following relation between the flow rates of a turbulent fluid containing polymers and an unforced laminar flow:

$$\langle U_l - \bar{V}_x \rangle = \frac{1}{P}[\langle (\overline{uw} - \langle \overline{uw} \rangle - \bar{\tau}_{xz} + \langle \bar{\tau}_{xz} \rangle)^2 \rangle + \langle |\nabla \boldsymbol{u}|^2 \rangle - \langle \boldsymbol{V} \cdot \boldsymbol{f} \rangle]. \qquad (4.8)$$

The first term in the right-hand side of the above equation represents the difference between the dissipation rate in the unforced laminar flow and the dissipation due to the mean flow in the case of a turbulent flow containing the polymer. The term $\langle |\nabla \boldsymbol{u}|^2 \rangle$ represents dissipation due to the fluctuations. In the case of a fluid containing polymers for drag reduction, the magnitude of $\langle \boldsymbol{V} \cdot \boldsymbol{f} \rangle$ will be small, since the required concentration of polymers is very low. Hence, the term $\langle |\nabla \boldsymbol{u}|^2 \rangle$ dominates the above equation, since the dissipation due to fluctuations is known to be significantly greater than that due to the mean flow in a turbulent fluid (Pope 2000). Any significant drag reduction will, therefore, be achieved by reducing $\langle |\nabla \boldsymbol{u}|^2 \rangle$.

It is now clear that for a polymer force to raise the bulk flow rate of the fluid to greater than or equal to that of a laminar Poiseuille flow, the following would need to hold:

$$\langle \boldsymbol{V} \cdot \boldsymbol{f} \rangle \geqslant \langle (\overline{uw} - \langle \overline{uw} \rangle - \bar{\tau}_{xz} + \langle \bar{\tau}_{xz} \rangle)^2 \rangle + \langle |\nabla \boldsymbol{u}|^2 \rangle. \qquad (4.9)$$

The question of whether or not the presence of a polymer force may produce sublaminar drag, therefore, becomes a question of the sign and magnitude of $\langle \boldsymbol{V} \cdot \boldsymbol{f} \rangle$. Hence, because of (3.1), we conclude that polymer forces cannot produce sublaminar drag in turbulent fluids.

Furthermore, by removing all of the fluctuating terms from (4.8), we obtain a relation between the bulk flow rate of a *laminar* Poiseuille flow subject to a polymer force to that of the equivalent unforced laminar flow. It is clear by inspection that if the polymer force is anywhere non-zero, then the bulk flow rate will be reduced. We may infer from this that while such polymer forces are known to be capable of causing drag reduction in turbulent fluids, they will invariably increase the drag when acting upon a laminar flow.

An alternative methodology, which could have been employed in this proof, has been presented by Bewley & Aamo (2004), who employed it in reference to drag reduction for a Poiseuille flow subject to blowing and suction-flow control at the walls. The difference between their methodology, and that employed here is that they have explicitly considered the magnitude of the drag at the wall. To rederive this result via their methodology would simply be a matter of removing all terms in Bewley (2009) which relate to the blowing and suction method of drag reduction, and adding a body force term, as we have done here, to account for the effect of the polymer.

5. Minimising drag

There is an alternative way to prove the result given in §4. By adding $P\langle \bar{V}_x \rangle$ to both sides of (2.15), and substituting (3.1), we obtain

$$\langle \bar{V}_x \rangle = \lim_{L \to \infty} \frac{1}{L^2} \int_{-1/2}^{1/2} \int_{-L/2}^{L/2} \int_{-L/2}^{L/2} \left(2\bar{V}_x - \frac{1}{P} \left| \frac{\mathrm{d}\bar{V}_x}{\mathrm{d}z} \right|^2 - \frac{1}{P} |\nabla \boldsymbol{u}|^2 \right) \mathrm{d}x\, \mathrm{d}y\, \mathrm{d}z - \langle \epsilon_p \rangle. \qquad (5.1)$$

While there is no unique solution to the above equation, it is possible, using the calculus of variations, to find the functions for $\bar{V}_x$ and $\boldsymbol{u}$ which maximise the value of the integral. To do so, we must first set $\langle \epsilon_p \rangle$ to zero. This is equivalent to assuming that the rate of dissipation of elastic energy within the polymer is negligible. We can, however, do this without loss of generality, since a non-zero $\langle \epsilon_p \rangle$ term would only ever reduce the value of $\langle \bar{V}_x \rangle$.

The derivation is omitted here, but the functions which maximise $\langle \bar{V}_x \rangle$ are

$$\bar{V}_x|_{max\langle \bar{V}_x \rangle} = \frac{P}{2}\left(\frac{1}{4} - z^2\right), \quad \boldsymbol{u}|_{max\langle \bar{V}_x \rangle} = \boldsymbol{0}, \tag{5.2}$$

which is the profile of an unforced laminar flow, $U_l(z)$. This implies that of all the physically allowable flow profiles of a Poiseuille fluid, the greatest bulk flow rate is obtained when flow profile is exactly that of an unforced laminar flow.

Therefore, since (4.6) indicates that the presence of a polymer force will cause a deviation to the profile of a laminar flow away from (5.2), we can conclude that adding such elastic polymers to a laminar flow will only result in increased drag.

6. Concluding remarks

The results show that the flow rate of an unforced laminar fluid constitutes an upper bound for the flow rate of a fluid containing dilute elastic polymers. This result can also potentially be applied to flows subject to drag reduction achieved by the addition of other types of particles to the fluid. Examples include the addition of surfactant molecules to a flowing liquid, and the addition of water droplets or sand particles to a flowing gas.

However, the additives which produce the drag reduction may also alter the fluid's density and viscosity, as is the case for elastic polymers. If their presence has affected the fluid's density or viscosity, then it is the flow rate of a laminar fluid with the same density and viscosity as the solution (rather than the density and viscosity of the pure solvent) which constitutes an upper bound on the bulk flow rate of the mixture.

Furthermore, if the resulting mixture produces a fluid that is either compressible or non-Newtonian, then this proof may not apply. However, as we have argued in § 1 of this paper, the minimum viscosity of a polymer solution will be greater than the viscosity of the pure Newtonian solvent.

An increased viscosity will reduce the flow rate. We can thus conclude that since the addition of the polymer cannot produce sublaminar drag when the thickening effect of the polymer is neglected, it will neither be capable of producing sublaminar drag in real flows in which the thickening effect of the polymer will often be significant.

By considering the overall energy equation (2.15) and the drag minimising procedure used in § 5, we see that for a body force (or boundary force) to produce sublaminar drag would require an energy input that is greater than the combination of the rate of dissipation due to turbulent fluctuations, $\langle |\nabla \boldsymbol{u}|^2 \rangle$, the dissipation within the polymer molecules, $\langle \epsilon_p \rangle$ and the effect of the deviation of the flow profile, $\bar{V}_x(z)$, away from its laminar equivalent, $U_l(z)$, caused by the Reynolds stress and the body force.

This concurs with Bewley's recent result (2009), showing that the power saved via sublaminar drag reduction produced by flow control must be less than the power cost to produce that drag reduction.

This method has not proved capable of deriving or approximating Virk's asymptote from first principles. The reason for this is that Virk's asymptote applies strictly to fully developed turbulent flow, for which the flow profile has also yet to be derived from first principles. Hence, we are only able to compare turbulent flows with drag reduction to laminar flow.

For this reason, we are only able to conclude that the drag is minimised by removing the fluctuations. However, by its definition, fully developed turbulent flow contains significant random fluctuations. At maximum drag reduction, these fluctuations will only have been *reduced* by the action of the polymer. Therefore, to derive Virk's asymptote would require the ability to quantify the amount by which these fluctuations may be reduced. This is something we cannot do without a theoretical basis for the nature of the fluctuations, and an analytical closure for the Navier–Stokes equation.

If the assumption of fully developed turbulent flow allowed some further assumptions to be made about the nature of the velocity vectors $\bar{\boldsymbol{V}}$ and $\boldsymbol{u}$, then it may prove possible to derive Virk's asymptote.

It is notable that the maximum drag reduction produced by the presence of surfactant micelles in the fluid is very similar to that produced by the presence of the polymer. In fact, the maximum drag reduction produced by the micelles slightly exceeds Virk's asymptote. It is also notable that when a flow containing a surfactant reaches maximum drag reduction, the Reynolds stress is everywhere zero (Warholic *et al.* 1999). (The Reynolds stress is given by $\overline{\boldsymbol{uu}}$. It first appears in the second term in left-hand side of (2.11).) The same is not true for a flow containing a polymer at the maximum drag reduction (Ptasinski *et al.* 2001).

The relevance of this is that the Reynolds stress represents the effect of the fluctuations upon the mean flow. It is via the Reynolds stress that the fluctuations draw energy from the mean flow. This transfer of energy is represented by the first term on the left-hand side of (2.13) and (2.14).

Any method of drag reduction, which does not act by directly imparting energy upon the mean flow, will function by altering the Reynolds stress. We have shown that no method of drag reduction which is due to the presence of a body force will be capable of producing sublaminar drag, unless its overall action imparts energy upon the flow. It follows that the net effect of a non-zero Reynolds stress in such flows must be to extract energy from the mean flow. Such methods of drag reduction, therefore, work by minimising the effect of the Reynolds stress upon the mean flow.

A flow with zero (or negligible) Reynolds stress, therefore, constitutes a natural upper bound for any such methods of drag reduction. In such a flow, the fluctuations can only extract energy from the mean flow indirectly via the body force. Such flows could then only differ in the extent to which the body force affects the mean flow.

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Appendix. Pipe flow

In this appendix, we extend these results to the common case of flow through a pipe. To do so, we use the cylindrical polar coordinates (r, θ, x), where x is the streamwise distance from the beginning of the pipe and r is the radial distance from the pipe's centre. A diagram of the system is shown in figure 2. The system's domain is

$$-\infty < x < \infty; \quad 0 \leqslant \theta < 2\pi; \quad 0 \leqslant r \leqslant 1. \tag{A 1}$$

The wall-parallel averages are again denoted by an overbar, and are now defined via

$$\bar{F}(r, t) \stackrel{\text{def}}{=} \lim_{L\to\infty} \frac{1}{L} \int_{-L/2}^{L/2} \int_0^{2\pi} F(r, \theta, x, t) \, \mathrm{d}\theta \, \mathrm{d}x, \tag{A 2}$$

and the overall average is denoted by angled brackets. It is now defined via

$$\langle F(t) \rangle \stackrel{\text{def}}{=} 2 \int_0^1 \bar{F}(r, t) r \, \mathrm{d}r. \tag{A 3}$$

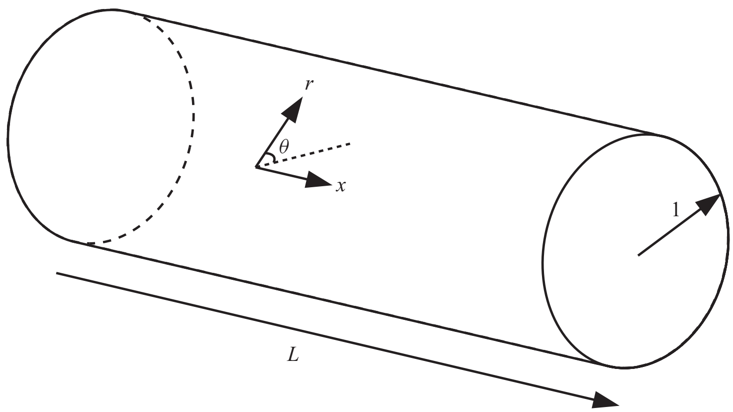

FIGURE 2. Diagram of the pipe domain. We consider an infinite pipe in which $L \to \infty$.

Again the Reynolds number, this time given by the pipe's radius and the bulk velocity, is $Re_B = \langle \bar{V}_x \rangle$. The pressure, velocity and polymer force are decomposed into mean and fluctuating components as before, this time with

$$V_x = \bar{V}_x + u, \quad V_\theta = v, \quad V_r = w. \tag{A 4}$$

The derivation follows a similar path to the channel-flow case, therefore, much of the detail will be omitted here. Beginning with the Navier–Stokes (2.1) and continuity (2.2) equations as before, we obtain the energy equations for a statistically steady-state flow via an entirely analogous path. The energy equations thus obtained are identical to those of channel flow (see (2.13)–(2.15)), except that the wall-normal direction z is here replaced by its pipe-flow analogue r.

We then proceed along the path described in § 4, in this way, we derive the following comparison between the flow rates of a turbulent fluid containing the polymer and an unforced laminar fluid:

$$\langle U_l - \bar{V}_x \rangle = \frac{1}{P}[\langle (\overline{uw} - \bar{\tau}_{xr})^2 \rangle + \langle |\nabla \boldsymbol{u}|^2 \rangle + \langle \epsilon_p \rangle], \tag{A 5}$$

which, for reasons already given, must be positive if $\overline{uw}$ is anywhere non-zero, or the polymer is present.

We may also prove this same result via the calculus of variations, as described in § 4. Since the energy equations are identical in form to those in the channel-flow case, we perform the calculus of variations upon an integral whose integrand is identical to that in (5.1), with the Cartesian coordinates used in the channel-flow case replaced by their cylindrical polar equivalents for the pipe flow. In doing so, we find that the flow profile which maximises the flow rate of a fluid in a pipe is,

$$\bar{V}_x|_{max\langle \bar{V}_x \rangle} = \frac{P}{4}(1 - r^2), \quad \boldsymbol{u}|_{max\langle \bar{V}_x \rangle} = \boldsymbol{0}, \tag{A 6}$$

which is the flow profile for an unforced laminar flow within a pipe.

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